Modeling Chess Rating Dynamics Using Differential Equations

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The Elo Rating Update Rule

The fundamental update for a player's rating after a game is:

$$R_{\text{new}} = R + K(S - E)$$

- R: current rating
- K: development coefficient (K-factor)
- S: actual game score ($1 =$ win, $0.5 =$ draw, $0 =$ loss)
- E: expected score, calculated by:

$$E = \frac{1}{1 + 10^{\frac{R_{\text{opp}} - R}{400}}}$$

Interpretation: E models the expected probability of winning based on rating difference.

From Discrete to Continuous Time

Players play many games over time, so ratings evolve continuously. We approximate discrete updates with a differential equation:

$$\frac{dR}{dt} = \lambda(t) \cdot K(t) \cdot [S(t) - E(R(t), R_{\text{opp}}(t))]$$

- $\lambda(t)$: frequency of games per unit time
- $K(t)$: development coefficient depending on experience
- $S(t)$: average actual score near time t
- $E(R, R_{\text{opp}})$: expected score formula as before

This models smooth rating changes over time.

Modeling Experience in Rating Changes

The USCF rating system modifies K based on experience:

$$K = \frac{800}{N' + m}$$

where

- N': number of games played
- m: minimum games before full stability (e.g., 5)

In continuous time, this generalizes to:

$$K(t) = \frac{800}{N(t)}, \quad N(t) = \int_0^t \lambda(\tau) d\tau$$

As the player plays more games ($N(t)$ increases), rating changes become smaller.

Rating Dynamics Against a Fixed Pool

For a player against a fixed opponent pool, the rating evolution follows:

$$\frac{dR}{dt} = \lambda(t) \cdot \frac{800}{N(t)} \cdot \left[S(t) - \frac{1}{1 + 10^{\frac{R_{\text{opp}} - R(t)}{400}}} \right]$$

This nonlinear ODE models rating adapting to performance and experience.

Modeling Two Players Competing

When two players $R_1(t)$ and $R_2(t)$ play only each other:

$$\frac{dR_1}{dt} = \lambda K \left[s_1 - \frac{1}{1 + 10^{\frac{R_2 - R_1}{400}}} \right]$$

$$\frac{dR_2}{dt} = \lambda K \left[s_2 - \frac{1}{1 + 10^{\frac{R_1 - R_2}{400}}} \right]$$

with

- s_1, $s_2 = 1 - s_1$: average scores per game
- λ, K: assumed constant here

This system captures the mutual rating influence in competition.

Finding Equilibrium Ratings

At equilibrium, ratings do not change over time:

$$\frac{dR_1}{dt} = 0, \quad \frac{dR_2}{dt} = 0$$

This gives the system:

$$s_1 = \frac{1}{1 + 10^{\frac{R_2 - R_1}{400}}}, \quad s_2 = \frac{1}{1 + 10^{\frac{R_1 - R_2}{400}}}$$

Since $s_2 = 1 - s_1$, these imply:

$$s_1 = E_1(R_1, R_2), \quad s_2 = 1 - s_1 = E_2(R_1, R_2)$$

Interpretation: At equilibrium, the actual average scores equal the expected scores based on ratings.

Solving for rating difference:

$$\frac{s_1}{s_2} = 10^{\frac{R_1 - R_2}{400}} \implies R_1 - R_2 = 400 \log_{10}\left(\frac{s_1}{s_2}\right)$$

This expresses the equilibrium rating gap in terms of the relative average scores.

Real-World application: Ding Liren (2728) vs Gukesh Dommaraju (2783), 2024 WCC

Initial ratings and parameters:

$$R_1 = 2728, \quad R_2 = 2783, \quad K = 10, \quad \lambda = 1$$

Calculate expected score for Ding:

$$E_1 = \frac{1}{1 + 10^{\frac{2783-2728}{400}}} = \frac{1}{1 + 10^{0.1375}} \approx 0.421$$

Actual average scores from match:

$$s_1 = \frac{6.5}{14} \approx 0.4643, \quad s_2 = \frac{7.5}{14} \approx 0.5357$$

Sources

- US Chess Federation. *The US Chess Rating System*.
  https://new.uschess.org/sites/default/files/media/documents/the-us-chess-rating-system.pdf
- *2700chess.com*.
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